Lűxue（律學）：The Chinese Scholarship on Musical Pitch and Tuning systems and Its Musicological and Acoustic Achievments

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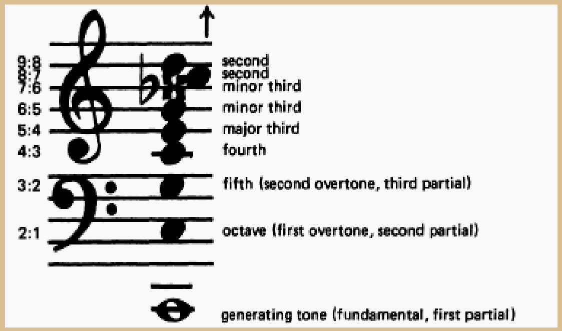
Summary

Introduction and Terminologies

■ *Lǘxue* 律學 or “The Study of *Lǘ*” in ancient Chinaan is an interdisciplinary research in art and science, which can roughly be translated as “the study of musical pitch and tuning systems” or “the study of tuning and teperament.” *Lǘ* means musical pitch within a tuning system.

■Some of the key terminologies in thie presentation:

1. **Tuning**
2. **Temperament**
3. **Comma**
4. **Cent**
5. **Harmonic Series**



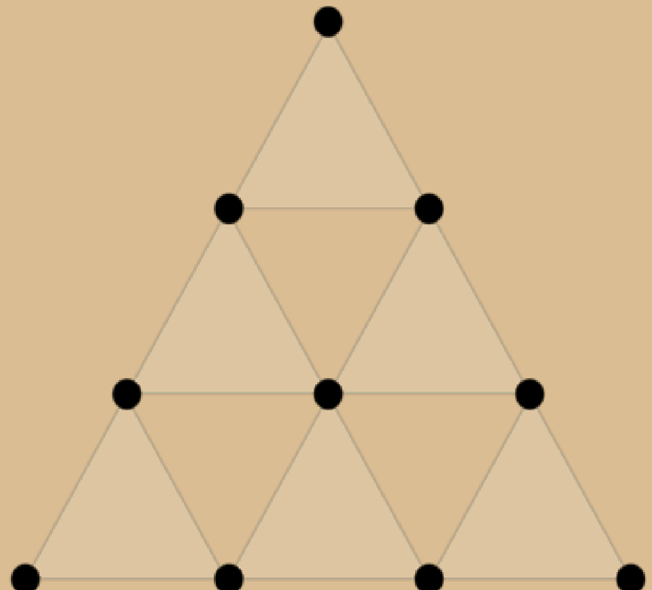
① Ancient Chinese understanding of musical pitches(*lű* 律) and Pythagoreansim

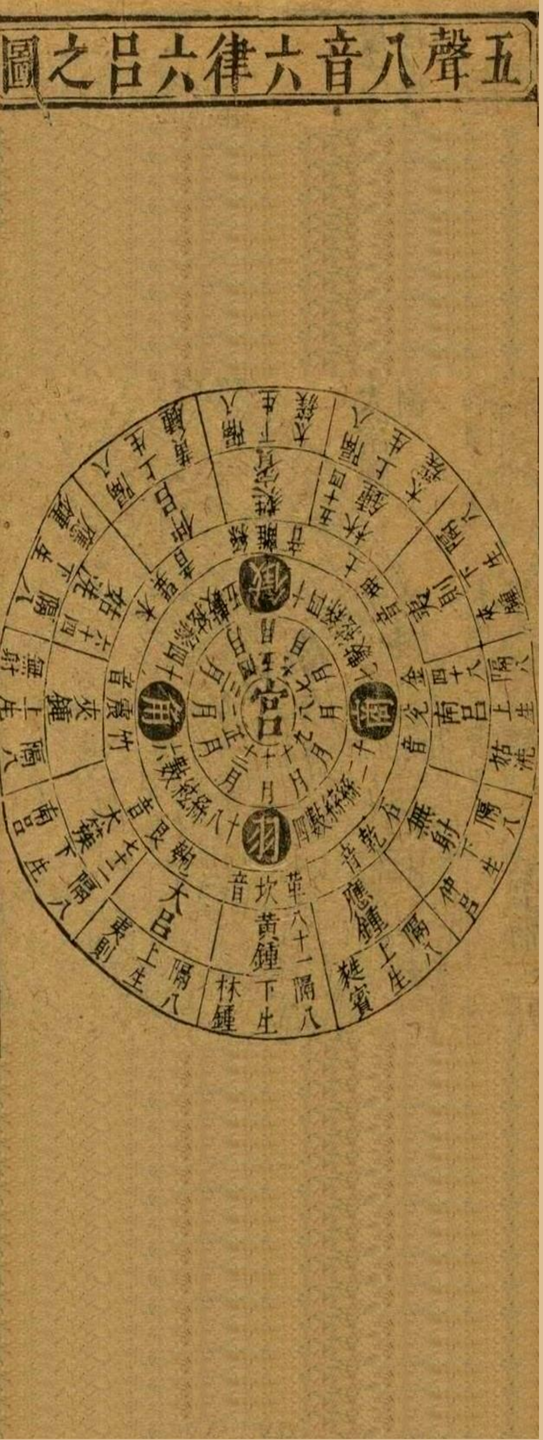
- It is well-known in the West that Pythagoras was the first person who discovered the mathematical nature of musical intervals.
- Pythagoreanism's most fundamental tenets include: numbers are constituent elements of reality, and numbers and their ratios provide the key to explaining the order of nature and the universe.

① Ancient Chinese understanding of musical pitches(*lǘ* 律) and Pythagoreansim

Pythagorean Tetractys

It is an arrangement of points in the shape of a triangle and represents the first four natural numbers, whose sum is 10 (1+2+3+4=10).





① Ancient Chinese understanding of musical pitches(*lǘ* 律) and Pythagoreansim

- In ancient China, one octave was divided into twelve pitches between half step for each, known as the twelve *lǘ*.
- *Lǘshi chunqiu* 呂氏春秋 ("Spring and Autumn Annals of Lǘ Buwei," 239 BCE) records the storey of Ling Lun 伶倫 creating the *lǘ* and made the *Gong* 宮(do) of *huangzhong*黃鐘 (C) its standard pitchin around 2,000 BCE. Although the archaeological discoveries prove the story would be somewhat exaggerated, the legend shows that the ancients Chinese had already realized that musical pitch could be preserved by pitch pipes.
- The term of *huangzhong* (“***yellow bell***”) in China was the name of the standard pitch, which is equivalent to today’s the “International standard pitch,” in which the A above “middle” C is tuned to 440 Hz.

① Ancient Chinese understanding of musical pitches(*lǘ* 律) and Pythagoreansim

1. *huangzhong* 黃鐘 (c),
2. *dalǘ* 大呂 (c#),
3. *taicu* 太簇 (d),
4. *jiazhong* 夾鐘 (d#),
5. *guxian* 姑冼 (e),
6. *zhonglǘ* 仲呂(e#),
7. *ruibin* 蕤賓(f#)
8. *linzhong* 林鐘 (g),
9. *yize* 夷則 (g#),
10. *nanlǘ* 南呂 (a),
11. *wuyi* 無射(a#),
12. *yingzhong* 應鐘 (b).

The earliest record of these names is found in the *Zhouyu* 周語 section of the *Guoyu* 國語 ('Chronicles of the States'), which was completed in the fifth century BCE.

In 1978, a set of *bianzhong* 編鐘 (bell set) was excavated in which the central part of the range, were divided into twelve semitones, which proves that the twelve *lǘ* recorded in *Guoyu* had in fact, already entered the musical practice of that period.

① Ancient Chinese understanding of musical pitches(*lű* 律) and Pythagoreansim

The study of musical pitch and tuning systems was always considered related to mathematics, the astronomical calendar, meteorology, philosophy of *yin-yang* and the five elements, and even political matters in ancient China. *Shangshu* 尚書 (“Esteemed Documents,” 1000 BCE) mentions “harmonizing time by adopting the right calendar, unifying the measurement of length, volume, and weight with the same musical pitch.”(“協時月正日，同律度量衡”).

① Ancient Chinese understanding of musical pitches(*lǘ* 律) and Pythagoreansim

When Wang Mang (王莽) usurped the imperial throne and established the short-lived Xin dynasty (新, 9–23 CE), he used sound as the foundation for imperial metrological standardization as part of his declaration of power. The ability to control sound was considered to reflect his ability to synchronize heaven and earth, the cosmic and the human. His authority and ruling legitimacy was considered rested in a single pitch pipe that produced the cardinal tone in the traditional musical system.

② *Sanfen Sunyi* 三分損益 (one-third reduction and addition) and *huangzhong huanyuan* 黃鐘還元 (*Huangzhong* Returns to Its Original Pitch)

Regarding how *lǘ* were produced, the earliest are found in *Guanzi* 管子by Guan Zhong 管仲 (ca. 645 BCE) of the Spring and Autumn Period, which explains how to produce *lǘ* by using the so-called“one-third reduction and addition” method.

② *Sanfen Sunyi* 三分損益 (one-third reduction and addition) and *huangzhong huanyuan* 黄鐘還元 (*Huangzhong* Returns to Its Original Pitch)

The method of "*sanfen sunyi*"

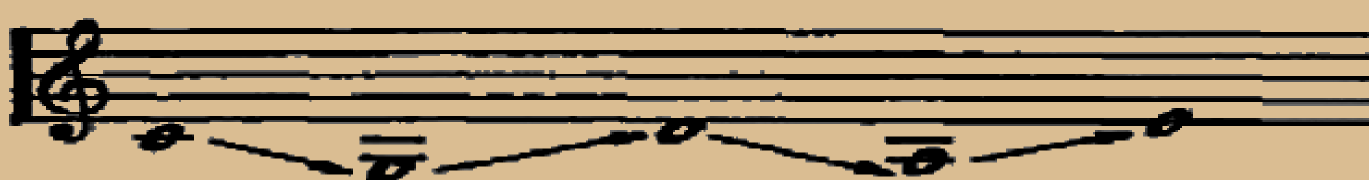
3^4=81 (*gong*, do)

("one-third addition) 81 X 4/3 - 108 (*zhi*, sol)

("one-third reduction") 108 X 2/3 = 72 (*shang*, re)

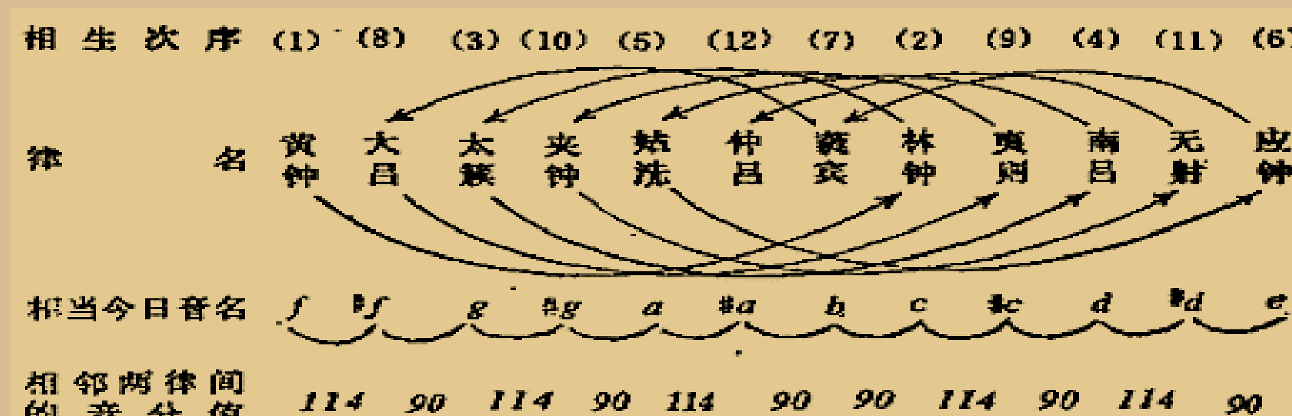
("one-third addition) 72 x 4/3 = 96 (*yu*, la)

("one-third reduction" 96 X 2/3 =64 (*jue*, mi)



② *Sanfen Sunyi* 三分損益 (one-third reduction and addition) and *huangzhong huanyuan* 黃鐘還元 (*Huangzhong* Returns to Its Original Pitch)

The book *Lǘshi Chunqiu* 呂氏春秋 (239 BCE), based on the five lǘ given in the *Guanzi*, continued to apply the *sanfen sunyi* method to produce the other seven *lǘ*.



② *Sanfen Sunyi* 三分損益 (one-third reduction and addition) and *huangzhong huanyuan* 黃鐘還元 (*Huangzhong* Returns to Its Original Pitch)

- The book *Huainan Zi* 淮南子, edited by Liu An (179-122 BCE) of the Han dynasty, established a separate *lű* number of 3^{11} = 177,147 for *huangzhong*. Then all other 11 pitches could expressed in round figures without using decimal points or fractions.

- Later on, the *Lűshu* 律書section of the *Shiji* 史記 ('Historical records') by Sima Qian (ca.145- 89 BCE) adopted the form of fractions, which is similar to that of the Greek Pythagorean tuning, which is why some scholars equate the two.

② *Sanfen Sunyi* 三分損益 (one-third reduction and addition) and *huangzhong huanyuan* 黃鐘還元 (*Huangzhong* Returns to Its Original Pitch)

Chinese Pitch Names	Method in *Guangzi* 管子, and *Lushi Hucnqiu* 吕氏春秋	Method in *shiji* 史記 (Historical Records)	Method in *Huainan Zi* 淮南子 (No Fraction)	Western Pitch Name
黃鐘huangzhong	81	1	3^{11}× 1 = 177,147	c
林鐘linzhong	81 * **2/3**= 54	2048/2187	177,147 × 4/3 =236,196	g
太簇taicu	54 * **4/3**= 72	8/9	236,196 × 2/3 = 157,464	d
南吕nanlǘ	72 * **2/3**= 48	16384/19683	157,464 × 4/3 =209,952	a
姑冼guxian	48 * **4/3**= 64	64/81	209,952 × 2/3 = 139,968	e
應鐘yingzhong	64 * **2/3**= 42.6667	131072/177147	139,968 × 4/3 = 186,624	b
蕤賓ruibin	42.6667 * **4/3**= 56.8889	512/729	186,624 × 2/3 = 124, 416	$f^{\#}$
大吕dalǘ	56.8889 * **4/3**= 75.8519	2/3	124,416x 4/3 = 165,888	$c^{\#}$
夷則yize	75.8519 * **2/3**= 50.5679	4076/6561	165,888X 4/3 = 221,184	$g^{\#}$
夾鐘jiazhong	50.5679 * **4/3**= 67.4239	16/27	221,184 × 2/3 =147,456	$d^{\#}$
無射wuyi	67.4239 * **2/3**= 44.9492	32768/59049	147,456 × 4/3 = 196,608	$a^{\#}$
仲吕zhonglǘ	44.9492 * **4/3**= 59.9323	128/234	196,608 X 2/3 = 131,072	$e^{\#}$
清黃鐘 qing huangzhong	[59.9323 * **2/3**= *39.9549*]		[131,072 x4/3 =174,762.6667]	b#

② *Sanfen Sunyi* 三分損益 (one-third reduction and addition) and *huangzhong huanyuan* 黃鐘還元 (*Huangzhong* Returns to Its Original Pitch)

In fact, the Chinese method resulted in ascending perfect fifths and descending perfect fourths. The Pythagorean tuning involves ascending perfect fifths or descending perfect fifths. So, six of the twelve *lǘ* between the two are similar, while the other six are different.

lü produced by OTRA method			Pythagorean method			
twelve *lü*	*lü* number	cent	twelve *lü*	*lü* number	cent	
Huangzhong (c)	1	0	c	1	0	
Dalü ($c^{\#}$)	$\frac{2048}{2187}$	114	d^{b}	$\frac{243}{256}$	90	✡
Taicu (d)	$\frac{8}{9}$	204	d	$\frac{8}{9}$	204	
Jiazhong ($d^{\#}$)	$\frac{16384}{19683}$	318	e^{b}	$\frac{27}{32}$	294	✡
Guxian (e)	$\frac{64}{81}$	408	e	$\frac{64}{81}$	408	
Zhonglü ($e^{\#}$)	$\frac{131072}{177147}$	522	f	$\frac{3}{4}$	498	✡
Ruibin ($f^{\#}$)	$\frac{512}{729}$	612	g^{b}	$\frac{729}{1024}$	588	✡
Linzhong (g)	$\frac{2}{3}$	702	g	$\frac{2}{3}$	702	
Yize ($g^{\#}$)	$\frac{4096}{6561}$	816	a^{b}	$\frac{81}{128}$	792	✡
Nanlü (a)	$\frac{16}{27}$	906	a	$\frac{16}{27}$	906	
Wuyi ($a^{\#}$)	$\frac{32768}{59048}$	1020	b^{b}	$\frac{9}{16}$	996	✡
Yingzhong (b)	$\frac{128}{243}$	1110	b	$\frac{128}{243}$	1110	

② *Sanfen Sunyi* 三分損益 (one-third reduction and addition) and *huangzhong huanyuan* 黃鐘還元 (*Huangzhong* Returns to Its Original Pitch)

The earliest theoretical account of Chinese music available in a European language is written by the Jesuit Joseph Amiot in Bejing in 1776 and published in Paris in 1780. Amiot considered that Chinese would have had a scale closely resembling the Pythagorean one more than eleven centuries before the birth of Pythagoras, and the Pythagorean claim for the invention of this scale was nothing less than an “act of robbery.”

② *Sanfen Sunyi* 三分損益 (one-third reduction and addition) and *huangzhong huanyuan* 黄鐘還元 (*Huangzhong* Returns to Its Original Pitch)

- The noted Chinese historian Zhu Qianzhi (朱謙之) considered the Greek system was more advanced in utilizing string length to do the calculation, while the original Chinese one was more primitive in using pitch pipes. So the Chinese system probably somehow spread to Greece and was developed there, but later on the Chinese learned from the Greeks when realizing the limitations of the pitch pipes in calculations.

- Joseph Needham, the author of *Science and Civilization in China*, proposed that "the simplest alternative hypothesis for which good reason can be found is that there radiated east and west from Babylonia the germ of an acoustic discovery which was developed in one way by the Greeks and in another by the Chinese...the Babylonians, who had many highly developed stringed instruments, would have made the observation."

② *Sanfen Sunyi* 三分損益 (one-third reduction and addition) and *huangzhong huanyuan* 黃鐘還元 (*Huangzhong* Returns to Its Original Pitch)

Musical Matters beyond the Practical Need

In ancient China, a year was divided into twelve lunar months, a day into twelve double hours, and an octave into twelve notes. These divisions were believed to be connected to societal order and rules. But scholars discovered that the movement of heavenly bodies all returned to their starting points, but a scale calculated using the *sanfen sunyi* (or Pythagorean) method did not.

② *Sanfen Sunyi* 三分損益 (one-third reduction and addition) and *huangzhong huanyuan* 黃鐘還元 (*Huangzhong* Returns to Its Original Pitch)

- It was Jing Fang of the Han Dynasty (77-37 BC) who first discovered that the cycle of the fifth did not return to the original *huangzhong* pitch as there was a discrepancy existing between starting and the ending notes.

$$\textit{zhishi}\ (b^{\#})\ 177{,}147 \times \left(\tfrac{2}{3}\right)^5 \times \left(\tfrac{4}{3}\right)^7 = 174{,}762\tfrac{2}{3};$$

$$\text{discrepancy: } 177{,}147 - 174{,}762\tfrac{2}{3} = 2{,}384\tfrac{1}{3}.$$

- In order to solve this problem, Jing Fang continued to use the *sanfen sunyi* method to produce up to the sixty *lǘ* in one octave. The number of the fifty-third *lǘ* in his system is very close to *huangzhong*, and this difference was called “one day” by Jing Fang, which was the smallest comma that human being ever understood during that time.

$$\left(\frac{3}{2}\right)^{53} \times \left(\frac{1}{2}\right)^{31} = \frac{3^{53}}{2^{84}}$$

$$3^{53}/2^{84} = 19383245667680019896796723/19342813113834066795298816$$

$$3.61504586553331404577968350811297 \approx 3.15 \text{ cents}$$

② *Sanfen Sunyi* 三分損益 (one-third reduction and addition) and *huangzhong huanyuan* 黃鐘還元 (*Huangzhong* Returns to Its Original Pitch)

During the Renaissance in the 17th century, Nicholas Mercator（1620 — 1687), William Holder (1616–1698), and Issac Newton, (1643—1727) also engaged in this area of research, and made similar discoveries. The so-called “53-tone equal temperament” and “Mercator’s comma” were all similar or identical discoveries to what Jing Fang had discovered more than 1700 years earlier, which were all based on the same mathematical formula.

$$\left(\frac{3}{2}\right)^{53} \times \left(\frac{1}{2}\right)^{31}$$

② *Sanfen Sunyi* 三分損益 (one-third reduction and addition) and *huangzhong huanyuan* 黃鐘還元 (*Huangzhong* Returns to Its Original Pitch)

A

TREATISE

OF THE

Natural Grounds, and *Principles*

OF

HARMONY.

By WILLIAM HOLDER, D. D. Fellow of the *Royal Society*, and late Sub-Dean of their MAJESTY's *Chapel Royal*.

To which is Added, by way of APPENDIX:

RULES for Playing a *Thorow-Bass*; with Variety of *Proper Lessons*, *Fuges*, and *Examples* to Explain the said RULES. Also Directions for Tuning an *Harpsichord* or *Spinnet*.

By the late Mr. GODFREY KELLER.

With several new Examples, which before were wanting, the better to explain some Passages in the former Impressions.

The whole being Revis'd, and Corrected from many gross Mistakes committed in the first Publication of these Rules.

LONDON:
Printed by W. PEARSON, over against *Wright's Coffee-House* in *Aldersgate-street*; for J. WILCOX in *Little-Britain*; and T. OSBORNE in *Gray's-Inn*. 1731.

Interval; let it be, How many are in *Diapason*? Which must be done by multiplying Comma's, *i. e.* adding them, till you arrive at a Ration equal to *Octave*, (if that be sought) *viz.* Duple: Or else by dividing the Ration of *Diapason* by that of a Comma, and finding the Quotient; which may be done by Logarithms. And herein I meet with some Differences of Calculations.

MERSENNUS finds, by his Calculation, $58\frac{1}{4}$ Comma's, and somewhat more, in an *Octave*: But the late *Nicholas Mercator*, a Modest Person, and a Learned and Judicious Mathematician, in a Manuscript of his, of which I have had a Sight, makes this Remark upon it; *In solvendo hoc Problemate aberrat Mersennus*: And he, working by the Logarithms, finds out but 55, and a little more; and from thence has deduced an ingenious Invention of finding and applying a least Common Measure to all Harmonic Intervals, not precisely perfect, but very near it.

SUPPOSING a Comma to be $\frac{1}{53}$ part of Diapason; for better Accommodation rather than according to the true Partition [illegible], which $\frac{1}{53}$, he calls an Artificial Comma, not exact, but differing from the true Na-

tural Comma about [illegible] part of a Comma, and [illegible] of Diapason (which is a Difference imperceptible) then the Intervals within Diapason will be measur'd by Comma's according to the following Table; which you may prove by adding two, or three, or more of these Numbers of Comma's, to see how they agree to constitute those Intervals, which they ought to make; and the like by substracting.

Intervals	$\frac{1}{53}$	Intervals	$\frac{1}{53}$
Comma	1	4^{th}	22
Diesis	2	*Tritone*	26
Semit. Minus	3	*Semidiapente*	27
Semit. Medium	4	5^{th}	31
Semit. Majus	5	6^{th} *Minor*	36
Semit. Maximus	6	6^{th} *Major*	39
Tone Minor	8	7^{th} *Minor*	45
Tone Major	9	7^{th} *Major*	48
3^{d} *Minor*	14	*Octave*	53
3^{d} *Major*	17		

THIS I thought fit, on this Occasion, to impart to the Reader, having Leave so to do from Mr. *Mercator*'s Friend, to whom he presented the said Manuscript.

HERE I may advertise the Reader, that it is indifferent whether you compare the greater

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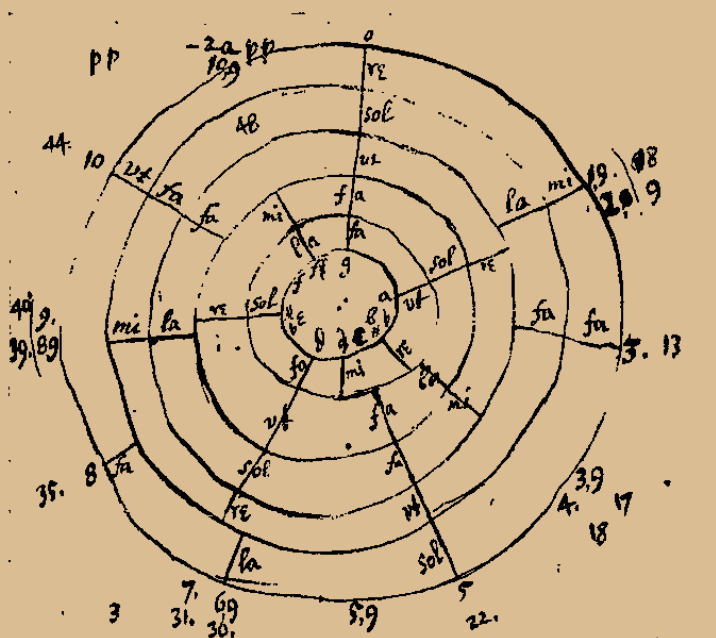
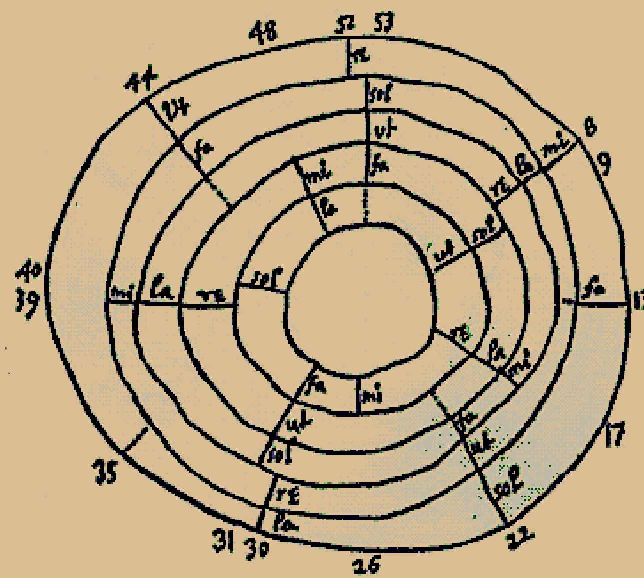
Issac Newton's Manuscripts Preserved in Oxford University's Library





③ Just Intonation Theory in the West and Its Practice in China

Before the end of the fifteenth century, a new type of monochord division was becoming popular, namely, divisions based on both the just fifth (2 : 3) and the just major third (4 :5). But the just intonation was known to be an inappropriate as a tuning system for keyboards.

③ Just Intonation Theory in the West and Its Practice in China

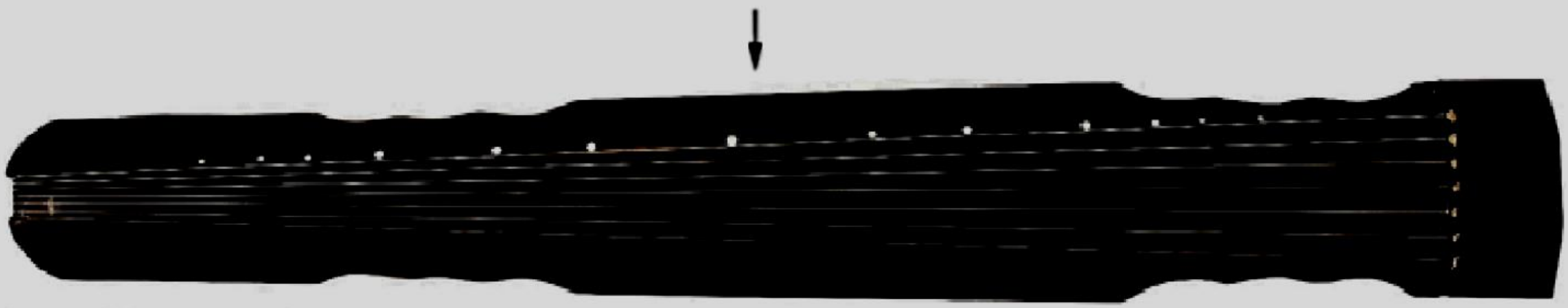
However, modern Chinese musicological research discovered that, the structure of the *guqin* zither, which was established in Han and Wei periods (206 - 265 BCE), made the Just Intonation possible in music practice since two thousand years ago, although no historical literature ever mentioned this tuning system because the *sanfen sunyi* was considered the only authentic and classical tuning system in the country.

③ Just Intonation Theory in the West and Its Practice in China

The 13 inlays position of the instrument surface.

FINGEOARD

position marks (*hui*)



Marker position	open string	13	12	11	10	9	8	7	6	5	4	3	2	1
Length Ratio	1	7/8	5/6	4/5	3/4	2/3	3/5	1/2	2/5	1/3	1/4	1/5	1/6	1/8
Stopped notes		231	316	386	498	702	884	1200	1586	1902	2400	2786	3102	3600
(octaves reduced)		231	316	386	498	702	884	±0	386	702	±0	386	702	±0
Harmonic notes		3600	310	278	240	1902	786	1200	2786	1902	2400	2786	3102	3600
(Reducing octaves)		±0	702	386	±0	702	386	±0	386	702	±0	386	702	±0

The ratio of the string length marked by the 13 inlays, and the pitches generated by the stopped and harmonic notes on a *guqin*.

④ Early Investigation of Equal Temperament: He Chengtian, Qian Yuesh, and Meantone Temperament

After Jing Fang's discovery, He Chengtian 何承天 (370-447) of the Song Period of the Southern Dynasties advocated making internal adjustments to the twelve *lǘ*, by dividing the difference of the diatonic comma into twelve parts and adding each part to each of the *lǘ* numbers.

④ Early Investigation of Equal Temperament: He Chengtian, Qian Yuesh, and Meantone Temperament

Comparision between He Chentian's New Tuning System and Equal Temperament

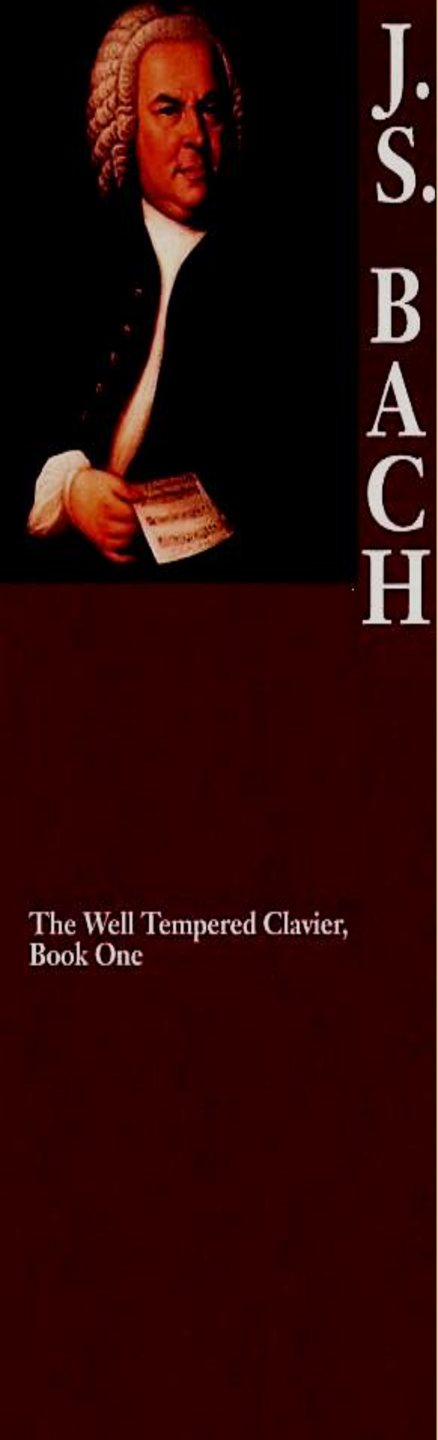
Name of *lü*	Method of calculating 'new *lü*'	Cents	Difference from equal temperament
Huangzhong (c)	$177,147 + 0 = 177,147$	0	± 0
Linzhong (g)	$177,147 \times \frac{2}{3} + 2,384\frac{1}{3} \times \frac{1}{12} = 118,296\frac{25}{36}$	699.04	− 0.96
Taicu (d)	$177,147 \times \frac{2}{3} \times \frac{4}{3} + 2,384\frac{1}{3} \times \frac{2}{12} = 157,861\frac{7}{18}$	119.55	− 0.45
Nanlü (a)	$177,147 \times (\frac{2}{3})^2 \times \frac{4}{3} + 2,384\frac{1}{3} \times \frac{3}{12} = 105,572\frac{1}{12}$	896.06	− 3.34
Guxian (e)	$177,147 \times (\frac{2}{3})^2 \times (\frac{4}{3})^2 + 2,384\frac{1}{3} \times \frac{4}{12} = 140,762\frac{7}{9}$	398.02	− 1.18
Yingzhong (b)	$177,147 \times (\frac{2}{3})^3 \times (\frac{4}{3})^2 + 2,384\frac{1}{3} \times \frac{5}{12} = 94,305\frac{17}{36}$	1091.44	− 8.56
Ruibin (f#)	$177,147 \times (\frac{2}{3})^3 \times (\frac{4}{3})^3 + 2,384\frac{1}{3} \times \frac{6}{12} = 125,608\frac{1}{6}$	595.22	− 4.78
Dalü (c#)	$177,147 \times (\frac{2}{3})^3 \times (\frac{4}{3})^4 + 2,384\frac{1}{3} \times \frac{7}{12} = 167,278\frac{31}{36}$	99.23	− 0.77
Yize (g#)	$177,147 \times (\frac{2}{3})^4 \times (\frac{4}{3})^4 + 2,384\frac{1}{3} \times \frac{8}{12} = 112,181\frac{5}{9}$	790.93	− 9.07
Jiazhong (d#)	$177,147 \times (\frac{2}{3})^4 \times (\frac{4}{3})^5 + 2,384\frac{1}{3} + \frac{9}{12} = 149,244\frac{1}{4}$	296.73	− 3.27
Wuyi (a#)	$177,147 \times (\frac{2}{3})^5 \times (\frac{4}{3})^5 + 2,384\frac{1}{3} \times \frac{10}{12} = 100,290\frac{17}{18}$	984.91	− 15.07
Zhonglü (e#)	$177,147 \times (\frac{2}{3})^5 \times (\frac{4}{3})^6 + 2,384\frac{1}{3} \times \frac{11}{12} = 133,257\frac{23}{36}$	492.87	− 7.13
Huangzhong (c)	$177,147 \times (\frac{2}{3})^5 \times (\frac{4}{3})^7 + 2,384\frac{1}{3} \times \frac{12}{12} = 177,147$	0	± 0

④ Early Investigation of Equal Temperament: He Chengtian, Qian Yuesh, and Meantone Temperament

During the same period in Song Yuanjia 元嘉 (424-453 CE) of the Southern Dynasties, Qian Lezhi 錢樂之continued to use *sanfen sunyi* method to produce up to 360 *lű* in one octave based on “Jing Fang's sixty *lű*” theory. The final pitch of the 360 *lű* was even closer to *huangzhong*'s high octave. The distance is even smaller than a schisma known later in the West.

④ Early Investigation of Equal Temperament: He Chengtian, Qian Yuesh, and Meantone Temperament

Huainan zi 《淮南子》 c. 122 B.C. Two-Digit Monochord					Jing Fang 京房 c. 45 B.C. 60- or 53-Division		He Chengtian 何承天 370-447 A.D. Linear Correction		Qian Lezhi 钱乐之 c. 450 A.D. 360-Division	
Pitch Name		Pitch	Cents							
Chinese	Western	Value	Huai	Pythagor.	Step No.	Cents	Tone	Cents	Step No.	Cents
Huang Chung	C	81	0	0	0	0	C	0	0	0
Lin Chung	G	54	702	702	48	93.84	C♯	101	154	101.07
T'ai Ts'u	D	72	204	204	2	203.91	D	200	255	198.53
Nan Lu	A	48	906	906	50	297.75	D♯	297	103	301.36
Ku Hsi	E	64	408	408	4	407.82	E	398	204	398.82
Ying Chung	B	43*	1096*	1110	52	501.66	E♯	493	358	499.89
Jui Pin	F♯	57	608	612	47	591.88	F♯	596	153	599.12
Ta Lü	C♯	76	110	114	1	701.96	G	699	307	700.19
I Tsê	G♯	51	801	816	49	795.80	G♯	791	102	799.41
Chia Chung	D♯	68	303	318	3	905.86	A	897	256	900.48
Wu I	A♯	45	1018	1020	51	999.70	A♯	985	51	999.70
Chung Lü	E♯	60	520	522	5	1109.78	B	1091	205	1100.78
Huang Chung	c			1224	53	1203.62	C'	1200	306	1198.23



④ Early Investigation of Equal Temperament: He Chengtian, Qian Yuesh, and Meantone Temperament

- After the Renaissance, the development of science and technology for musical instrument making made people realize the problems in both the Pythagorean tuning and the Just Intonation. The meantone temperament came into use.
- The name “meantone” was applied to this temperament because the tone, as C-D, is precisely half of the pure third, as C-E. One of the most famous applications using meantone temperament is the work of The Well-Tempered Clavier, two sets of preludes and fugues in all 24 major and minor keys for keyboard by Johann Sebastian Bach.

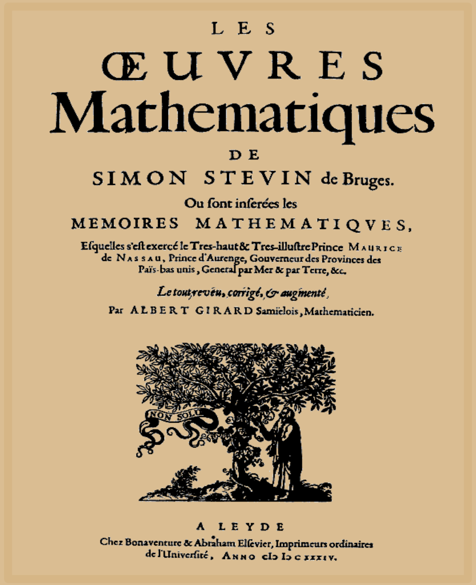
Tuning	Pitches							
	C	D	E	F	G	A	B	C
Pythagorean	204	204	90	204	204	204	90	
Just	204	182	112	204	182	204	112	
Meantone-Tempered	193	193	117	193	193	193	117	
Equal-Tempered	200	200	100	200	200	200	100	

⑤ Equal Temperament: Prince Zhu Zaiyu and Simon Stevin

The calculation of tempered intervals in the West was first performed towards the end of the sixteenth century by the Dutch mathematician and engineer Simon Stevin (1548–1620).

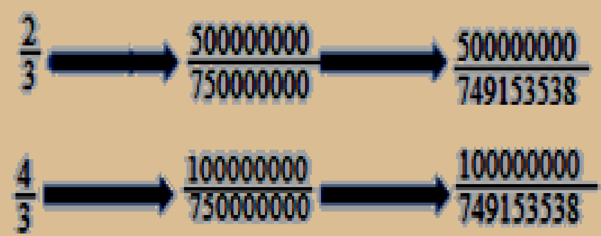
Stevin realized that the string lengths of an equal-tempered monochord required root extraction. In this case, he used a combination of one cube root and two square roots to perform the extraction of

$$\sqrt[12]{2}$$


LES
ŒUVRES
Mathematiques
DE
SIMON STEVIN de Bruges.
Ou sont inserées les
MEMOIRES MATHEMATIQVES,
Esquelles s'est exercé le Tres-haut & Tres-illustre Prince MAURICE de NASSAU, Prince d'Aurenge, Gouverneur des Provinces des Païs-bas unis, General par Mer & par Terre, &c.
Le tout revû, corrigé, & augmenté,
Par ALBERT GIRARD Samielois, Mathematicien.
A LEYDE
Chez Bonaventure & Abraham Elsevier, Imprimeurs ordinaires de l'Université. ANNO cIↃ IↃ c xxxiv.

⑤ Equal Temperament: Prince Zhu Zaiyu and Simon Stevin

- Meanwhile in China, by the sixteenth century, the Ming dynasty scientist Prince Zhu Zaiyu (1536-1611) finally completed a mathematical calculation of the twelve-tone equal temperaments. He called his method *xinfa milii* 新法密率 ('New method tight rate').
  - **One-third reduction**
  - **One-third addigion**



- Keeping the original sequence to produce *lǘ*, Zhu finally achieved the result of *zhonglǘ* returning to the original starting *huangzhong* pitch.

⑤ Equal Temperament: Prince Zhu Zaiyu and Simon Stevin

Comparasion between the results by Prince Zhu Zaiyu and Simon Stevin

$$\sqrt[12]{2} = 1.059463094359295264561825$$

律名	比率
正黄钟	1.000000000000000000000000
倍应锺	1.059463094359295264561825
倍无射	1.122462048309372981433533
倍南吕	1.189207115002721066717500
倍夷则	1.259921049894873164767211
倍林锺	1.334839854170034364830832
倍蕤宾	1.414213562373095048801689
倍仲吕	1.498307076876681498799281
倍姑洗	1.587401051968199474751706
倍夹锺	1.681792830507429086062251
倍太蔟	1.781797436280678609480452
倍大吕	1.887748625363386993283826
倍黄钟	2.000000000000000000000000

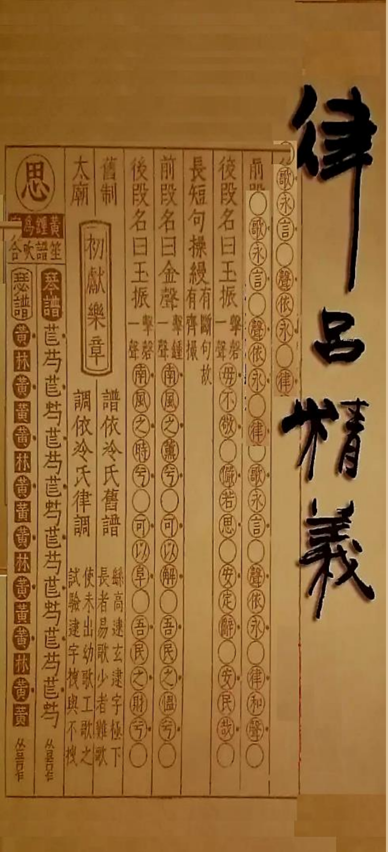
10 000	Selftone	First
9 438	Semitone	Minor second
8 909	Whole tone	Major second
8 409[1])	One-tone-and-half . . .	Minor third
7 936	Ditone	Major third
7 491	Two-tone-and-half . . .	Good fourth
7 071	Tritone	Bad fourth
6 674	Three-tone-and-half . . .	Fifth
6 298	Four-tone	Minor sixth
5 944	Four-tone-and-half . . .	Major sixth
5 611	Five-tone	Minor seventh
5 296	Five-tone-and-half . . .	Major seventh
5 000	Six-tone	Double-first
4 719	Six-tone-and-half	Double minor second
4 454	Seven-tone	Double major second

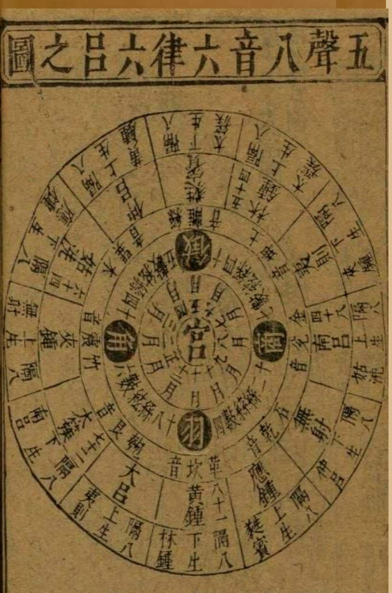
If one now wants to see how far amiss were the erroneous divisions of Pythagoras, Boëthius, and Zarlino, this is readily possible by putting the largest number of their ratio also 10 000. I take the Pythagorean division, whose table being described up to the three-tone-and-half, runs as follows:

---

[1]) In the table a mistake, 8404, has been corrected to 8409. The correct numbers should read:

10 000.0		
9 438.7	7.491.5	5 946.0
8 909.0	7 071.1	5 612.3
8 409.0	6 674.2	5 297.2
7 937.0	6 299.0	5 000.0





⑤ Equal Temperament: Prince Zhu Zaiyu and Simon Stevin

Although the detailed mathematical process of how Zhu reached the *xinfa milǘ* 新法密率 by extracting the 12th square root of 2 is still not clear. We know that like Stevin, Zhu also used the combination of a cube foot and two square roots in his extraction.

The date of Zhu's preface to *Lǘli Rongtong*律暦融通 ("Harmonising the *Lǘ*" 1581), was February 6th 1581, which shows that Zhu completed his mathematical calculation of the twelve-tone equal temperaments before that date.

⑤ Equal Temperament: Prince Zhu Zaiyu and Simon Stevin

Zhu Zaiyu's equal temperament calculation yielded the first mathematical realization of the 12-tone equal temperament in the world, which is at least 4 years earlier than the similar work done by Simon Stevin whose discovery, according to himself, was attributed to the concept of “proportion” in the Dutch language which did not exist in Greek. In Stevin’s treatise, he listed each interval from the pitches of “first” to “double first” chromatically to finally equate each half step.

Nevertheless, in ancient China, the relationship between the central pitch and others were deemed important, but the relationships among the subordinating notes themselves were not, and concept of interval is vague.

⑤ Equal Temperament: Prince Zhu Zaiyu and Simon Stevin

- Joseph Needham speculated that Zhu's discovery might have spread to the West before Stevin’s work did, probably even by word of mouth from a traveler. But it is doubtful given the different motivations and approaches betwen the two of them as I mentioned befoare.
- Nevertheless, although Zhu and Stevin's research were completed in the opposite sides of the world, it reflected the tireless human endeavors in this interdisciplinary field of research between art and science.

Summary

The research of the mathematical and acoustic laws in music were systematically recorded in official histories in ancient China, where scientists and music scholars continued to resolve the problem of how to close the cycle of the fifth caused by the “diatonic comma” or “Pythagorean Comma” on metaphysical level.

From any monographs on the history of music theory today, we would glean the work by scientists who made outstanding contributions resolving some of the music problems of the past. By inheriting the ancient scholastic tradition of integrating knowledge learning in both Art and Science, we could revitalize tremendous human creativity in the new age of the 21st century, and spartk a new phenomen in our education, technology, artistic creativity, and entrepreneurial industries.

Thank you very Much!